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Spacecraft Orientation Based on Space Object Observations by Means of Quaternion Algebra

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The problem of spacecraft orientation through observations of space objects having zero as well as nonzero daily parallax is solved by means of quaternion algebra. It is assumed that there are unit vectors, which are known directions of several space objects in the inertial reference system, and the same unit vectors are measured in the body-fixed reference system. A new quaternion algorithm is suggested for the case when measurements of "near-body" objects are used. The solution of the problem is formulated as a quaternion conditional extremum problem, and both the position vector and three-axis attitude of the spacecraft are determined by the Lagrange multipliers method. The application of quaternions avoids numerous trigonometric calculations and gives a solution suitable for both theoretical analysis and computations. The method can also be useful in space photogrammetry and astronomy.

Nomenclature

A = quaternions and matrices (uppercase letters)
A^T = transposed matrix
a = scalar (lowercase letters)
$\boldsymbol{a}$ = column vector
I_n = identity ($n \times n$) matrix

I. Introduction

THE present paper contains an application of quaternions to the well-known spacecraft orientation problem that can be summarized as follows. Assume that there are several space objects with unit vectors of their directions defined in an inertial reference frame. These space objects can be the sun, planets, stars, satellites, etc. On the other hand, coordinates of the same unit vectors are measured in the spacecraft–body coordinate system. Having this set of the vector measurements, one has to determine the position vector and attitude of the spacecraft in the inertial reference frame. Two different cases are possible: The observed objects have zero and nonzero daily parallax. It means that the distance between the spacecraft and the observed objects is considered to be infinitely large in comparison with the distance between the spacecraft and the origin of the inertial reference frame in the first case, and the distances are of the same order in the second case. For example, if the spacecraft is moving around the Earth or the sun, one has the case of zero daily parallax objects if stars are observed and the case of nonzero daily parallax if the observed objects are planets or satellites. In the first case the unit vectors in the inertial and in the body-fixed reference frames can be considered colinear. This case is nothing else but the well-known Wahba's problem.[1] Various deterministic and optimal algorithms based on matrix–vector and quaternion representations were suggested and thoroughly examined both from theoretical and numerical points of view. See, for instance, Lerner,[2] Shuster,[3–6] Markley,[7,8] Golub and von Matt,[9] Grienger,[10] Bar-Itzhack and Oshman,[11] and many others. In photogrammetry the same problem is known as the rototranslation problem, and its solution, which is closest to the ideas of the present paper, was given by Sanso.[12]

The case of nonzero daily parallax objects is more complicated for a description since the unit vectors are now not colinear but coplanar. Existing matrix–vector algorithms are rather burdensome and they call for numerous trigonometric calculations. See, for instance, Urmaev.[13] We suggest a new solution based on quaternion representation for this case.

The paper consists of two parts. In the first one the well-known principal properties of quaternion algebra as well as new results in this field are given briefly for the reader's sake. We used the book by Branez and Shmiglevsky,[14] the paper by Grafarend and Schaffrin,[15] the paper by Battin,[16] and the paper by the author[17] when expounding this theory.

The second part is devoted to the precise statement of the problem and its solution in quaternions. Our main concern is with objects having nonzero daily parallax. The solution is formulated as a conditional extremum problem in the set of quaternions and both the position vector and three-axis attitude of the spacecraft are determined. We use the same quaternion variant of the Lagrange multipliers method, which was sucessfully employed in the previous papers by the author for the exterior photograph orientation problem,[17] the relative photograph orientation problem,[18] and problems of the Kalman filter.[19,20] Two cases are considered: attitude of the spacecraft is known or unknown. An exact solution is given in the first case and a linearization of quaternion equations is suggested in the second one.

II. Some Knowledge of Quaternion Algebra

A. General Observations Concerning Quaternions

The quantity $A = a_0 + ia_1 + ja_2 + ka_3$, where a_0, a_1, a_2, a_3 are any real numbers and i, j, k are so-called imaginary units, is

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called a quaternion. (In some papers the notation $A = ia_1 + ja_2 + ka_3 + a_4$ is used.) The numbers a_0, a_1, a_2, a_3 are called coefficients of A. If $a_0 = 0$, the quaternion is called an imaginary one. The imaginary units i, j, k can be identified with basis vectors of a three-dimensional frame of reference S; hence a_1, a_2, a_3 can be considered as coordinates of a certain vector $\boldsymbol{a}$ in S. It is symbolically written as $A = a_0 + \boldsymbol{a}$. The scalar (or real) part of A is denoted as $a_0 = \text{Re } A$ and the vector (or imaginary) part of it is $\boldsymbol{a} = \text{Im } A$.

On the other hand, an imaginary quaternion R can be associated with any vector $\boldsymbol{r}$ by the rule $R = ir_1 + jr_2 + kr_3$, where r_1, r_2, r_3 are coordinates of $\boldsymbol{r}$ in a frame of reference S. We use a subscript, as in R_S, to denote the frame of reference where $\boldsymbol{r}$ is considered.

Two quaternions are equal to each other if their real and imaginary parts are equal correspondingly.

Arithmetic operations are carried out as follows:

$$A + B = a_0 + b_0 + \boldsymbol{a} + \boldsymbol{b}, \qquad cA = ca_0 + c\boldsymbol{a} \tag{1}$$

where c is a real number. These operations are commutative, associative, and distributive.

The product of two quaternions A and B is defined as

$$AB = a_0 b_0 - \boldsymbol{a} \cdot \boldsymbol{b} + a_0 \boldsymbol{b} + b_0 \boldsymbol{a} + \boldsymbol{a} \times \boldsymbol{b} \tag{2}$$

where $\boldsymbol{a} \cdot \boldsymbol{b}$ is a scalar product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ and $\boldsymbol{a} \times \boldsymbol{b}$ is their vector product. It is clear that quaternion multiplication is associative, distributive, but noncommutative. Evidently, $AB = BA$ if and only if either one of the quaternions is a scalar or their vector parts are colinear vectors. However, we have, for any quaternion,

$$\text{Re } AB = \text{Re } BA \tag{3}$$

Equality (3) makes it possible to transpose quaternions in spite of their multiplication being noncommutative.

The following relations emerge from Eq. (2) for imaginary quaternions:

$$\begin{aligned} AB + BA &= -2\boldsymbol{a} \cdot \boldsymbol{b} = 2 \text{ Re } AB \\ AB - BA &= 2\boldsymbol{a} \times \boldsymbol{b} = 2 \text{ Im } AB \\ C(AB - BA) + (AB - BA)C &= -4\boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{c} \\ &= 4 \text{ Re } (C \text{ Im } AB) \end{aligned} \tag{4}$$

where $\boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{c}$ is a scalar triple product of vectors. Thus the well-known vector relations can be written in a quaternion form as relations between the associated quaternions:

$$\begin{aligned} AB + BA &= 0 \quad \text{(orthogonality)} \\ AB - BA &= 0 \quad \text{(collinearity)} \\ C(AB - BA) + (AB - BA)C &= 0 \quad \text{(coplanarity)} \end{aligned} \tag{5}$$

The following important assertion can be easily proved:

$$\text{If Re } AB = 0 \text{ for all } B, \text{ then } A = 0 \tag{6}$$

The quaternion $A^* = a_0 - \boldsymbol{a}$ is called the conjugate quaternion. Thus $\text{Re } A^* = \text{Re } A$, $\text{Im } A^* = -\text{Im } A$. The following properties of A^* hold:

$$\begin{array}{ll} 1)\ (A^*)^* = A & 4)\ A + A^* = 2 \text{ Re } A \\ 2)\ (A + B)^* = A^* + B^* & 5)\ A - A^* = 2 \text{ Im } A \\ 3)\ (AB)^* = B^* A^* & 6)\ A^* = -A \text{ iff Re } A = 0 \end{array} \tag{7}$$

The norm of quaternion A is $\|A\|^2 = a_0^2 + |\boldsymbol{a}|^2$, where $|\boldsymbol{a}|$ is the length of $\boldsymbol{a}$. If $\|A\| = 1$, the quaternion is called normalized. The following properties of the norm are obvious:

$$\begin{array}{ll} \|A\| = 0 \text{ iff } A = 0, & \|A\|^2 = AA^* = A^*A \\ \|A\| = \|A^*\|, & \|AB\| = \|A\|\|B\| \end{array} \tag{8}$$

A reciprocal quaternion A^{-1} is defined for $A \neq 0$ as $A^{-1}A = 1$. For normalized quaternions $A^{-1} = A^*$.

Every quaternion A can be also represented in a trigonometric form

$$A = \|A\|(\cos u + \boldsymbol{e} \sin u) \tag{9}$$

where $\cos u = a_0/\|A\|$, $\sin u = |\boldsymbol{a}|/\|A\|$, $0 \leq u < \pi$, and the unit vector $\boldsymbol{e} = \boldsymbol{a}/|\boldsymbol{a}|$. The vector $\boldsymbol{e}$ is called an axis of A.

B. Relations Between Quaternions and Rotations

The application of quaternions to a description of three-dimensional rotations is based on the following main theorems.

Theorem 1. Let A and R be quaternions with nonzero vector parts where A is represented by Eq. (9). Then the norm and the scalar part of $R' = ARA^{-1}$ are equal to those of R. The imaginary part $\text{Im } R'$ is equal to the vector resulting from rotation of $\text{Im } R$ through the angle $2u$ about the axis $\boldsymbol{e}$.

The most important statement of quaternion theory is that a set of normalized quaternions is isomorphic to a set of three-dimensional rotations. This is a consequence of Theorem 1.

If a frame of reference S is rotated to a new position S' and a normalized quaternion A corresponds to this rotation, this is denoted as

$$S' = ASA^* \tag{10}$$

Notation (10) means that quaternions $R_S(\boldsymbol{e}_q)$ and $R_S(\boldsymbol{e}'_q)$, associated with the basis vectors $\boldsymbol{e}_q$ and $\boldsymbol{e}'_q$, $q = 1, 2, 3$ of S and S', respectively, are related to each other as follows:

$$R_S(\boldsymbol{e}'_q) = AR_S(\boldsymbol{e}_q)A^*, \qquad q = 1, 2, 3 \tag{11}$$

We used A^* in Eqs. (10) and (11) because $A^{-1} = A^*$ for normalized quaternions.

The conjugate quaternion A^* realizes the inverse rotation.

Theorem 2. Let $\boldsymbol{r}$ be a fixed vector in a reference frame S and R_S be its associated quaternion. Let a new reference frame S' be obtained by formula (10). Then the new coordinates of $\boldsymbol{r}$ with respect to S' coincide with the coefficients of $\text{Im } R_{S'}$, where $R_{S'} = A^* R_S A$.

Theorem 2 is the inverse to Theorem 1. The latter describes the changing of vector coordinates under a rotation of this vector and Theorem 2 corresponds to a reference frame rotation.

Up to this point we considered only rotation of the reference frames. Now, assume that the rotation is supplemented with a parallel displacement. Let S and S' be three-dimensional right-handed orthogonal frames of reference as before. Here, S' is rotated about S by formula (10) and then is translated in parallel. Let $\boldsymbol{b}$ be a position vector of the origin of S' with respect to S. Assume that $\boldsymbol{r}$ is an arbitrary vector in S. According to Theorem 2, the coordinates of $\boldsymbol{r}$ with respect to S' coincide with the coefficients of the imaginary part of $R_{S'} = A^* R_S A$, where R is a quaternion associated with $\boldsymbol{r}$. At the same time, if $\boldsymbol{r}$ is treated as a position vector of its terminal point M, the coordinates of this immobile point M with respect to S' are the coefficients of the imaginary part of quaternion $R_{S'} = A^*(R_S - B_S)A$, where B_S is a quaternion associated with $\boldsymbol{b}$. Of course, when constructing quaternions R_S and B_S, both $\boldsymbol{r}$ and $\boldsymbol{b}$ must be related to S.

C. Relations Between Quaternions and Matrices

Now, let us denote a four-dimensional vector with coordinates equal to the coefficients of a quaternion A by $\boldsymbol{v}_A$. Using Eq. (2), coordinates of the vector $\boldsymbol{v}_{AB}$ corresponding to the product AB can be written in the matrix form

$$\boldsymbol{v}_{AB} = G_1(A)\boldsymbol{v}_B = G_2(B)\boldsymbol{v}_A \tag{12}$$

where matrices $G_1(A)$ and $G_2(B)$ are the following:

$$G_1(A) = \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix}$$
$$G_2(B) = \begin{bmatrix} b_0 & -b_1 & -b_2 & -b_3 \\ b_1 & b_0 & b_3 & -b_2 \\ b_2 & -b_3 & b_0 & b_1 \\ b_3 & b_2 & -b_1 & b_0 \end{bmatrix} \tag{13}$$

For an arbitrary number of factors one has

$$\boldsymbol{v}_{A_1A_2\cdots A_n} = G_1(A_1)\cdots G_1(A_{n-1})\boldsymbol{v}_{A_n} = G_2(A_n)\cdots G_2(A_2)\boldsymbol{v}_{A_1} \tag{14}$$

Using Eq. (12) it can be shown that there are the following relations between coordinates of $\boldsymbol{v}_{A_1A_2\cdots A_n}$ and coordinates of $\boldsymbol{v}_{A_m}$, $m = 2, \ldots, n-1$:

$$\begin{aligned} \boldsymbol{v}_{A_1\cdots A_n} &= G_1(A_1)\cdots G_1(A_{m-1})G_2(A_n)\cdots G_2(A_{m+1})\boldsymbol{v}_{A_m} \\ &= G_2(A_n)\cdots G_2(A_{m+1})G_1(A_1)\cdots G_1(A_{m-1})\boldsymbol{v}_{A_m} \end{aligned} \tag{15}$$

The following properties of matrices G_1 and G_2 are immediately deduced from Eqs. (12–15):

$$\begin{aligned} &1)\ G_q(A^*) = G_q^T(A), \qquad q = 1, 2 \\ &2)\ G_q(A+B) = G_q(A) + G_q(B), \qquad q = 1, 2 \\ &3)\ G_1(AB) = G_1(A)G_1(B), \qquad G_2(AB) = G_2(B)G_2(A) \\ &4)\ G_1(A)G_2(B) = G_2(B)G_1(A). \end{aligned} \tag{16}$$

From Eq. (15) and property 1 of Eq. (16) it follows that

$$\boldsymbol{v}_{ABA^*} = G_1(A)G_2^T(A)\boldsymbol{v}_B \tag{17}$$

If quaternion B has a zero scalar part, then Eq. (17) is reduced to the equalities

$$\begin{aligned} &\operatorname{Re} ABA^* = 0 \\ &\operatorname{Im} ABA^* = \left[\left(2a_0^2 - \|A\|^2\right)I_3 + 2\boldsymbol{a}\boldsymbol{a}^T + 2a_0K(\boldsymbol{a})\right]\operatorname{Im} B \end{aligned} \tag{18}$$

where $K(\boldsymbol{a})$ is a skew-symmetric matrix defined by

$$K(\boldsymbol{a}) = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \tag{19}$$

It is easy to verify that for a normalized quaternion A the matrix on the right of relation (18) is nothing but the well-known Rodriguez matrix.

By definition, one has that $K(\boldsymbol{a})$ is the cross-product matrix, i.e., $K(\boldsymbol{a})\boldsymbol{r} = \boldsymbol{a} \times \boldsymbol{r}$. Hence, $K(\boldsymbol{a})\boldsymbol{a} = 0$ and $K(\boldsymbol{a})\boldsymbol{r} = -K(\boldsymbol{r})\boldsymbol{a}$.

The relationships between quaternions and matrices considered above can be applied to solving a linear quaternion equation. A general form is

$$AX + XB + CXD = F \tag{20}$$

where X is an unknown quaternion and A, B, C, D, F are known ones. Transferring Eq. (20) to matrix form, one has

$$[G_1(A) + G_2(B) + G_1(C)G_2(D)]\boldsymbol{v}_X = \boldsymbol{v}_F \tag{21}$$

This matrix equation has a unique solution if and only if the determinant of the system does not vanish.

D. Extremum Problems in Set of Quaternions

Various problems connected with rotations of bodies can be formulated as extremum problems in a set of quaternions; i.e., one has to find a point of extremum of a loss function $f(X)$, where $f(X)$ is a scalar function of a quaternion argument X. Thus, we need to have a suitable method to determine this point of extremum.

We define the directional derivative of the scalar function $f(X)$ by the following way. Let H be an arbitrary quaternion and ε be a scalar quantity; the derivative of $f(X)$ in the direction of H is

$$f'_H(X) = \lim_{\varepsilon\to 0}\frac{\partial}{\partial\varepsilon}f(X + \varepsilon H) \tag{22}$$

Using results of functional calculus,[21] it can be proved that if the continuous function $f'_H(X)$ of quaternion argument X exists for every H, then that point X in which the equality

$$f'_H(X) = 0 \tag{23}$$

is fulfilled for every H is a point of an extremum. Moreover, if $f(X)$ is a quadratic function, this point is a point of a minimum.

Condition (23) contains an arbitrary quaternion H, but it is possible to eliminate it for the whole class of quadratic loss functions by using the properties of quaternions. It is the most important class for practical problems, namely,

$$f(X) = \sum_{q=1}^{n}\|F_q(X)\|^2 \tag{24}$$

where $F_q(X)$ is a quaternion function of a quaternion argument. Since this result is of main importance for future considerations let us consider a simple example to make it clear.

Example. One has to find the minimum of the function $f(X) = \|A - BX\|^2$, where A, B are known quaternions. According to definition (22), using also Eq. (3) and the properties of conjugate quaternions, one has

$$\begin{aligned} f'_H(X) &= \lim_{\varepsilon\to 0}\frac{\partial}{\partial\varepsilon}\|A - B(X + \varepsilon H)\|^2 \\ &= \lim_{\varepsilon\to 0}\frac{\partial}{\partial\varepsilon}(A - BX - \varepsilon BH)(A - BX - \varepsilon BH)^* \\ &= -BH(A - BX)^* - (A - BX)(BH)^* \\ &= -2\operatorname{Re} BH(A - BX)^* = -2\operatorname{Re}(A - BX)^*BH \end{aligned} \tag{25}$$

As it must be zero for every H, using Eq. (6), one gets immediately an algebraic quaternion equation $(A - BX)^*B = 0$. Hence, $X = B^{-1}A$, and it is a point of a minimum evidently.

In the multivariable case when $f(X_1, X_2, \ldots, X_n)$ is a scalar function of several quaternion arguments, the directional derivative with respect to X_q is defined as

$$f'_{X_q,H}(X_1, X_2, \ldots, X_n) = \lim_{\varepsilon\to 0}\frac{\partial}{\partial\varepsilon}f(X_1, \ldots, X_q + \varepsilon H, \ldots, X_n) \tag{26}$$

where H is an arbitrary quaternion and ε is a scalar as before. In a point of an extremum we have the set of equations, for every H,

$$\begin{aligned} &f'_{X_1,H}(X_1, \ldots, X_n) = 0 \\ &\vdots \\ &f'_{X_n,H}(X_1, \ldots, X_n) = 0 \end{aligned} \tag{27}$$

It is possible to eliminate H from Eq. (27) for the quadratic loss multivariable functions of the same type as Eq. (24).

Other examples of extremum problems in sets of quaternions are described in papers by the author.[17–20]

It is not possible to give here an exhaustive review of all properties of quaternions. Hence we restrict our considerations to the ones given above.

III. Problem of Spacecraft Orientation Through Observations of Space Objects

In this section, a general problem of spacecraft orientation through observation (or photographs) of several space objects is considered. We do not make a distinction between "to photograph" and "to observe," implying that the unit direction vectors of the objects observed or, what is the same, of their images in the photograph are related to a body-fixed reference frame.

A. Formulation of Problem in Terms of Quaternions

Let us consider the following reference systems: S is a three-dimensional, right-handed, and orthogonal body-fixed reference frame and G is an arbitrary, three-dimensional, right-handed orthogonal inertial reference frame. The position and attitude of the spacecraft are identified with the position of the origin of S and attitude of S with respect to G.

Two types of observed objects are possible. One type (stars) has zero daily parallax, since stars are usually assumed to be infinitely distant objects. The other type (planets, satellites, etc.) is assumed to have nonzero daily parallax. The choice of G is dictated by the specific character of the problem. For example, the origin of G can be chosen in the center of mass of a planet system when planets are observed. It also seems to be useful to take G to be an inertial frame of reference with the origin at the barycenter of the Earth–moon system when satellites are used for orientation. Finally, if one has star observations, the choice of G is governed by reasons of convenience.

Let r_q, $q = 1, 2, \ldots, n$, be unit vectors fixing directions to the objects observed with respect to G and $\boldsymbol{m}_q$ be unit vectors fixing directions to the objects in S. One has to determine the coordinates of the origin of S (vector $\boldsymbol{b}$) related to G and the attitude of S with respect to G at the instant of observation.

In terms of quaternions we have the following formulation of the problem. Let R_{qG} and M_{qS} be imaginary quaternions corresponding to $\boldsymbol{r}_q$ and $\boldsymbol{m}_q$, respectively. The subscripts G and S indicate the reference frame to which the vector is related. Let X be an unknown quaternion describing the rotation that connects S and G and B be a quaternion associated with the unknown vector $\boldsymbol{b}$ related to G.

Taking into account only rotation, we have

$$G = XSX^* \tag{28}$$

If S is a body-fixed photograph reference frame, one has to use a scale factor in Eq. (28) additionally because of the difference between units of measurement in S and G. We omit this factor for simplicity. Otherwise all considerations are readily corrected.

Since X describes only rotation, it is a normalized quaternion. Relationship (28) between G and S dictates the relationship between quaternions M_{qS} and M_{qG} that are associated with $\boldsymbol{m}_q$ as follows:

$$M_{qG} = X^* M_{qS} X \tag{29}$$

Then two cases have to be considered.

1) *Case of objects with zero daily parallax.* In this case, the directions of $\boldsymbol{r}_q$ and $\boldsymbol{m}_q$ coincide. These vectors are collinear with norms equal to 1 (Fig. 1). Hence, the coordinates of the vectors are equal to each other when the vectors are related to a common reference frame. Imaginary quaternions M_{qG} and R_{qG} must be equal too:

$$R_{qG} = X^* M_{qS} X \tag{30}$$

The error of this equality is an imaginary quaternion $Q_q(X) = R_{qG} - X^* M_{qS} X$. According to the method of least squares, one has to minimize the scalar function $f(X)$ of a quaternion argument

$$f(X) = \sum_{q=1}^{n} \|Q_q(X)\|^2 \tag{31}$$

with respect to X under the condition $\|X\|^2 = 1$.

It is easy to see that the formulation obtained above is the quaternion form of Wahba's problem (or rototranslation problem in photogrammetry). The unknown quaternion B is absent in Eq. (31). It means that the moderate parallel displacement has no effect on star observations and only the spacecraft attitude X can be determined in this case.

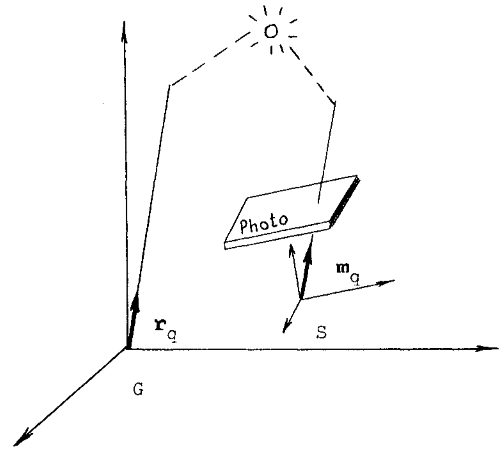

Fig. 1 Geometry of collinearity condition (case of zero daily parallax).

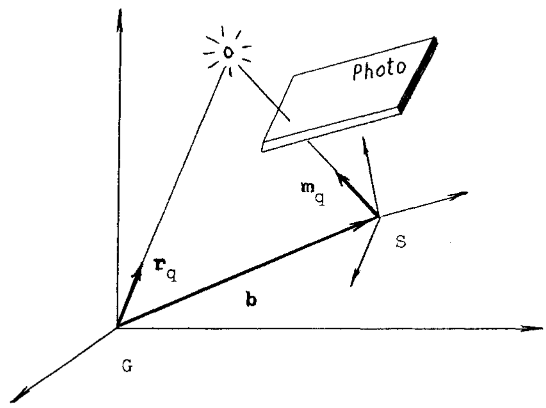

Fig. 2 Geometry of coplanarity condition (case of nonzero daily parallax).

2) *Case of objects with nonzero daily parallax.* Now, $\boldsymbol{r}_q$ and $\boldsymbol{m}_q$ are not collinear, but there is a coplanarity condition that can be used. This condition implies that the two unit vectors $\boldsymbol{r}_q$ and $\boldsymbol{m}_q$ as well as the unknown vector $\boldsymbol{b}$ must lie in a common plane (Fig. 2). The coplanarity condition is also fundamental for the relative orientation problem in photogrammetry when one has to reconstruct the perspective conditions between a pair of photographs that existed at the instant of exposure. (See Ref. 18.) The coplanarity condition means that the scalar triple product of $\boldsymbol{r}_q$, $\boldsymbol{m}_q$, and $\boldsymbol{b}$ is zero for any q when these vectors are related, for instance, to G:

$$\boldsymbol{r}_q \times \boldsymbol{m}_q \cdot \boldsymbol{b} = 0 \tag{32}$$

According to Eq. (5), this relationship written in quaternion form is

$$(R_{qG} M_{qG} - M_{qG} R_{qG}) B + B (R_{qG} M_{qG} - M_{qG} R_{qG}) = 0 \tag{33}$$

Then, substituting Eq. (29) into Eq. (33) and denoting

$$T_q(X) = R_{qG} X^* M_{qS} X - X^* M_{qS} X R_{qG} \tag{34}$$

one gets the basic relationship

$$T_q(X) B + B T_q(X) = 0 \tag{35}$$

The quaternion $T_q(X)$ has a simple geometric significance. Clearly its vector part $\tau_q = \operatorname{Im} T_q(X)$ is twice a vector product of $\boldsymbol{r}_q$ and $\boldsymbol{m}_q$ when both are related to G, and the scalar part of $T_q(X)$ is equal to zero, that is, $T_q^* = -T_q$.

The error equation corresponding to Eq. (35) is

$$e_q(X, B) = T_q(X) B + B T_q(X) \tag{36}$$

The quantity e_q is not a quaternion but a scalar, as it is equal to twice the scalar product of b and τ_q. According to the method of least squares, we have to minimize the scalar function of the quaternion arguments

$$\phi(X, B) = \sum_{q=1}^{n} \|e_q(X, B)\|^2 \tag{37}$$

with respect to X and B under the condition

$$\|X\|^2 = 1 \tag{38}$$

In spite of e_q being a scalar, we write $\|e_q\|^2$ instead of e_q^2 in order to be more general for future considerations.

Evidently, $\phi(X, B)$ has a minimum for $B = 0$, even if there is a distance between the origins of S and G, that is, $B \neq 0$. It means that we must have an additional condition for getting a nonzero solution. Suppose, for instance, that the length of the vector b is measured at the instant of observation, i.e.,

$$|b| = \|B\| = \beta \tag{39}$$

Condition (39), as well as Eq. (38), is necessary for the mathematical treatment of the quaternion extremum problem.

Thus, the initial problem is reduced to the conditional extremum problem in terms of quaternions.

B. Solution of Conditional Extremum Problem

The conditional extremum problem (37–39) is solved by the Lagrange multipliers method. Let $f(X, B, \gamma_1, \gamma_2)$ be an auxiliary Lagrange function

$$f(X, B, \gamma_1, \gamma_2) = \phi(X, B) + \gamma_1(\|X\|^2 - 1) + \gamma_2(\|B\|^2 - \beta^2) \tag{40}$$

where γ_1 and γ_2 are the scalar Lagrange multipliers. The partial derivative of $f(X, B, \gamma_1, \gamma_2)$ with respect to X in the direction of an arbitrary quaternion H is

$$f'_{X,H}(X, B, \gamma_1, \gamma_2) = \lim_{\varepsilon \to 0} \frac{\partial}{\partial \varepsilon} \times \left(\sum_{q=1}^{n} \|e_q(X + \varepsilon H, B)\|^2 + \gamma_1 \|X + \varepsilon H\|^2 \right) \tag{41}$$

Since e_q depends on the quaternion T_q, the latter one will be our initial concern. Using Eq. (34) and retaining only the linear terms in ε, one has the following representation of T_q when X is changed to $X + \varepsilon H$:

$$T_q(X + \varepsilon H) = T_q(X) + \varepsilon L_q(X, H) + \cdots \tag{42}$$

where

$$L_q(X, H) = R_{qG} H^* M_{qS} X + R_{qG} X^* M_{qS} H - H^* M_{qS} X R_{qG} - X^* M_{qS} H R_{qG}$$

Next, according to Eq. (36) and also retaining only the linear terms in ε, we have

$$e_q(X + \varepsilon H, B) = e_q(X, B) + \varepsilon(L_q B + B L_q) + \cdots \tag{43}$$

Then, the derivative of $e_q(X, B)$ is

$$[e_q(X, B)]'_{X,H} = 2 \operatorname{Re} e_q(X, B)(L_q B + B L_q) \tag{44}$$

Bearing in mind that e_q is a scalar and hence can be interchanged with the symbol Re, one has finally

$$f'_{X,H}(X, B, \gamma_1, \gamma_2) = 2 \sum_{q=1}^{n} e_q(X, B) \operatorname{Re}(L_q B + B L_q) + 2\gamma_1 \operatorname{Re} X H^* \tag{45}$$

Now, we need to reduce this expression to a form that allows use of assertion (6) to eliminate the arbitrary quaternion H. We transform Eq. (45) by using the following considerations. Evidently, from Eq. (42), $L_q^* = -L_q$ insofar as M_{qS} and R_{qG} are imaginary quaternions. Thus

$$L_q B + B L_q = 2 \operatorname{Re} L_q B \tag{46}$$

On substituting Eq. (42) for L_q into the right side of Eq. (46), we then transform every item in it. Using properties of conjugate quaternions and equality (3), we obtain

$$\begin{aligned}
\operatorname{Re} R_{qG} H^* M_{qS} X B &= \operatorname{Re} M_{qS} X B R_{qG} H^* \\
\operatorname{Re} R_{qG} X^* M_{qS} H B &= \operatorname{Re} H B R_{qG} X^* M_{qS} = -\operatorname{Re} M_{qS} X R_{qG} B H^* \\
\operatorname{Re} H^* M_{qS} X R_{qG} B &= \operatorname{Re} M_{qS} X R_{qG} B H^* \\
\operatorname{Re} X^* M_{qS} H R_{qG} B &= -\operatorname{Re} M_{qS} X B R_{qG} H^*
\end{aligned} \tag{47}$$

Combining similar terms, we have

$$\operatorname{Re} L_q B = 2 \operatorname{Re}(M_{qS} X B R_{qG} - M_{qS} X R_{qG} B) H^* \tag{48}$$

By this means one has an ultimate expression for the partial derivative:

$$f'_{X,H}(X, B, \gamma_1, \gamma_2) = 8 \sum_{q=1}^{n} e_q(X, B) \operatorname{Re}(M_{qS} X B R_{qG} - M_{qS} X R_{qG} B) H^* + 2\gamma_1 \operatorname{Re} X H^* \tag{49}$$

As it must be zero for an arbitrary quaternion H, according to Eq. (6), we get the first algebraic equation

$$4 \sum_{q=1}^{n} e_q(X, B) M_{qS} X (B R_{qG} - R_{qG} B) + \gamma_1 X = 0 \tag{50}$$

It does not contain H any more.

Analogously, the partial derivative with respect to B is

$$f'_{B,H}(X, B, \gamma_1, \gamma_2) = \lim_{\varepsilon \to 0} \frac{\partial}{\partial \varepsilon} \times \left(\sum_{q=1}^{n} \|e_q(X, B + \varepsilon H)\|^2 + \gamma_2 \|B + \varepsilon H\|^2 \right) \tag{51}$$

Substituting Eq. (43) into Eq. (51) and rearranging, one has

$$f'_{B,H}(X, B, \gamma_1, \gamma_2) = 4 \sum_{q=1}^{n} e_q(X, B) \operatorname{Re} T_q^* H^* + 2\gamma_2 \operatorname{Re} B H^* \tag{52}$$

This derivative must be zero for an arbitrary quaternion H too; hence we obtain the second algebraic equation

$$2 \sum_{q=1}^{n} e_q(X, B) T_q^* + \gamma_2 B = 0 \tag{53}$$

Thus, we have obtained the set of four algebraic quaternion equations (38), (39), (50), and (53) with respect to the unknown quaternions X, B and scalar multipliers γ_1, γ_2.

Let us further consider two cases: Orientation of the spacecraft is known and unknown at the instant of measurements.

C. Case of Known Orientation

When the orientation of the spacecraft is known, the quaternion X is known, and only B (i.e., vector b) must be determined. If there were no instrumental errors, b might be easily found from the following simple considerations.

As r_q, m_q, and b are coplanar, b is orthogonal to the vector product of r_q and m_q. Hence, b is orthogonal to τ_q, and we have the following equations for b:

$$\tau_q^T b = 0, \qquad q = 1, 2, \ldots, n \tag{54}$$

Coordinates of τ_q are constructed directly from the data r_q, m_q, X in the following way. First, using Eq. (18), we find the coordinates of m_q in G:

$$m_{qG} = \operatorname{Im} X^* M_{qS} X = \left[\left(2x_0^2 - 1\right) I_3 + 2xx^T - 2x_0 K(x)\right] m_q \quad (55)$$

where $x_0 = \operatorname{Re} X$, $x = \operatorname{Im} X$. Then, by the definition of matrix K, one has

$$\tau_q = 2K(r_q) m_{qG} \quad (56)$$

It follows immediately from Eq. (54) that b cannot be uniquely determined if only one object is observed, as there is a whole plane orthogonal to the only vector τ_1. When there are just two objects observed, b is uniquely expressed in the form

$$b = (\tau_1 \times \tau_2 / |\tau_1 \times \tau_2|)\beta \quad (57)$$

The direction of b should be taken such that it lies in the part of space where the spacecraft is located.

Finally, if there are more than two observed objects, the set of equations (54) is undetermined and one has to use the method of least squares in a regular way, i.e., to minimize the loss function with respect to b:

$$\sum_{q=1}^{n} \left(\tau_q^T b\right)^2 \quad (58)$$

The same situation occurs whenever the measurement errors exist.

However, we can find b without the foregoing well-known considerations but at once from the quaternion equation (53). Using Eq. (36), Eq. (53) becomes

$$2\sum_{q=1}^{n} \left(T_q B T_q^* + B\|T_q\|^2\right) + \gamma_2 B = 0 \quad (59)$$

According to Eq. (18), we have $\operatorname{Re} T_q B T_q^* = 0$ and $\operatorname{Im} T_q B T_q^* = (-\|T_q\|^2 I_3 + 2\tau_q \tau_q^T) b$ as $\operatorname{Re} T_q = 0$. Then in terms of matrices, Eq. (59) becomes

$$4\sum_{q=1}^{n} \tau_q \tau_q^T b + \gamma_2 b = 0 \quad (60)$$

Apparently the desired vector b and the scalar γ_2 are respectively an eigenvector and an eigenvalue of the symmetric matrix $\sum_{q=1}^{n} \tau_q \tau_q^T$. The eigenvector is determined within an arbitrary constant factor that can be used to satisfy the equality $|b| = \beta$. As $\sum_{q=1}^{n} \tau_q \tau_q^T$ is a real symmetric matrix of dimension 3 × 3, it has three real eigenvalues that can be determined exactly. The desired eigenvalue γ_2 is the one such that the value $-\gamma_2\beta^2/4$ is the smallest. To verify this condition, multiply Eq. (60) by b^T on the left, yielding

$$4\sum_{q=1}^{n} (\tau_q^T b)^2 + \gamma_2 \beta^2 = 0 \quad (61)$$

When Eq. (61) is compared with Eq. (58), the eigenvalue requirement is confirmed.

The comparison of the known results and the solution in quaternions cited in this paragraph verify the correctness of the quaternion formulas.

D. Case of Unknown Orientation of Spacecraft

The complete set of Eqs. (38), (39), (50), and (53) has to be solved in this case. It is not a linear problem; hence we have first to linearize it.

For simplicity of notation, we shall omit further indices S and G as there is no risk of ambiguity. Of course, we keep in mind that quaternions M_q and R_q are associated with m_q and r_q measured in S and G, respectively. For the sake of simplicity, assume also that m_q and r_q are measured exactly.

Let X_0, B_0 be initial approximations of the unknown quaternions X and B, and $\|X_0\| = 1$, $\|B_0\| = \beta$. The initial approximations of unknown scalars γ_1 and γ_2 can be found in the following way. On multiplying Eq. (50) by X^* to the left, one has

$$4\sum_{q=1}^{n} e_q(X, B) X^* M_q X (B R_q - R_q B) + \gamma_1 = 0 \quad (62)$$

After substituting X_0 and B_0 in Eq. (62), one gets the initial value γ_{10}. Analogously, on multiplying Eq. (59) by B^* to the left, one has

$$2\sum_{q=1}^{n} \left(B^* T_q B T_q^* + \beta^2 \|T_q\|\right) + \gamma_2^2 \beta^2 = 0 \quad (63)$$

The initial value γ_2 is found from Eq. (63) when $X = X_0$, $B = B_0$.

Preparatory to linearize our set of quaternion equations, we will first consider the X_0 dependence of T_q. Let δB, δX, $\delta\gamma_1$, $\delta\gamma_2$ be the desired corrections. On substituting X_0 and δX for X and εH, respectively, in Eq. (42) and retaining only the linear terms, one has

$$T_q(X_0 + \delta X) = T_q(X_0) + R_q\, \delta X^* M_q X_0 + R_q X_0^* M_q\, \delta X$$
$$- X_0^* M_q\, \delta X R_q - \delta X^* M_q X_0 R_q = T_{q0} + \delta T_q \quad (64)$$

where $T_{q0} = T_q(X_0)$. In terms of vectors, Eq. (64) becomes

$$v_{T_q} = v_{T_{q0}} + v_{\delta T_q}$$

In view of Eq. (15) the components of $v_{\delta T_q}$ depend on those of δX as follows:

$$v_{\delta T_q} = (G_1(R_q) G_2(M_q X_0) - G_2(M_q X_0 R_q)) v_{\delta X^*}$$
$$+ [G_1(R_q X_0^* M_q) - G_1(X_0^* M_q) G_2(R_q)] v_{\delta X} \quad (65)$$

Since $\delta X^* = \operatorname{Re}\delta X - \operatorname{Im}\delta X$, $v_{\delta X^*} = J v_{\delta X}$, where

$$J = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad (66)$$

Next, using properties 1–4 of G_1 and G_2, after evident rearrangements, Eq. (65) becomes

$$v_{\delta T_q} = D_q v_{\delta X} \quad (67)$$

where

$$D_q = [G_1(R_q) - G_2(R_q)][G_1(X_0^* M_q) + G_2(M_q X_0) J]$$

The properties of G_1 and G_2 imply also that the first row of the matrix D_q contains only zeros for every q. It is simple to check the calculations after constructing D_q.

Then, using Eq. (36) and also retaining only the linear terms,we have, for $e_q(X, B)$,

$$e_q(X_0 + \delta X, B_0 + \delta B) = e_q(X_0, B_0) + \delta T_q\, B_0 + T_{q0}\, \delta B$$
$$+ B_0\, \delta T_q + \delta B\, T_{q0} = e_{q0} + \delta e_q \quad (68)$$

or, in terms of vectors,

$$v_{e_q} = v_{e_{q0}} + v_{\delta e_q} \quad (69)$$

The vector $v_{\delta e_q}$ is given as

$$v_{\delta e_q} = [G_1(B_0) + G_2(B_0)] v_{\delta T_q} + [G_1(T_{q0}) + G_2(T_{q0})] v_{\delta B}$$
$$= [G_1(B_0) + G_2(B_0)] D_q v_{\delta X} + [G_1(T_{q0}) + G_2(T_{q0})] v_{\delta B} \quad (70)$$

Finally, denoting $Y_q = B R_q - R_q B$, we have

$$Y_q(B_0 + \delta B) = Y_q(B_0) + \delta B\, R_q - R_q\, \delta B = Y_{q0} + \delta Y_q \quad (71)$$

where $Y_{q0} = Y_q(B_0)$. In terms of matrices Eq. (71) becomes

$$v_{Y_q} = v_{Y_{q0}} + v_{\delta Y_q} = v_{Y_{q0}} + [G_2(R_q) - G_1(R_q)] v_{\delta B} \quad (72)$$

Now, the main set of equations is easily linearized. Retaining only linear terms, Eq. (53) becomes

$$P_{10} + 2\sum_{q=1}^{n}(e_{q0}\delta T_q^* + \delta e_q T_{q0}^*) + \gamma_{20}\,\delta B + \delta\gamma_2 B_0 = 0 \tag{73}$$

where P_{10} means the left-hand side of Eq. (53) when all unknowns take initial values. In matrix form, using Eqs. (67) and (70) and considering also that $v_{\delta T_{q*}} = J v_{\delta T_q}$, Eq. (73) becomes

$$D_{11}v_{\delta B} + D_{12}v_{\delta X} + v_{B0}\,\delta\gamma_2 = -v_{P10} \tag{74}$$

where

$$D_{11} = 2\sum_{q=1}^{n} G_2(T_{q0}^*)[G_1(T_{q0}) + G_2(T_{q0})] + \gamma_{20}I_4$$

$$D_{12} = 2\sum_{q=1}^{n}[G_2(T_{q0}^*)[G_1(B_0) + G_2(B_0)] + e_{q0}J]D_q$$

It can be easily shown by using the properties of G_1 and G_2 that the first rows of matrices D_{11} and D_{12} involve only zeros. On the one hand, it is also a suitable condition for checking the calculations when constructing these matrices and, on the other hand, it points to the fact that matrix Eq. (74) consists of only three ordinary equations.

Analogously, Eq. (50) after linearization becomes

$$P_{20} + 4\sum_{q=1}^{n}(\delta e_q M_q X_0 Y_{q0} + e_{q0}M_q\,\delta X\,Y_{q0} + e_{q0}M_q X_0\,\delta Y_q)$$
$$+\,\delta\gamma_1 X_0 + \gamma_{10}\,\delta X = 0 \tag{75}$$

where P_{20} means the left-hand side of Eq. (50) when all unknowns take initial values.

Then, omitting intermediate rearrangements, one re-expresses Eq. (75) in terms of matrices,

$$D_{21}v_{\delta B} + D_{22}v_{\delta X} + \delta\gamma_1 v_{X0} = -v_{P20} \tag{76}$$

where

$$D_{21} = 4\sum_{q=1}^{n}\{G_2(M_q X_0 Y_{q0})[G_1(T_{q0}) + G_2(T_{q0})]$$
$$+\,e_{q0}G_1(M_q X_0)[G_2(R_q) - G_1(R_q)]\}$$

$$D_{22} = 4\sum_{q=1}^{n}\{G_2(M_q X_0 Y_{q0})[G_1(B_0) + G_2(B_0)]D_q$$
$$+\,e_{q0}G_1(M_q)G_2(Y_0)\} + \gamma_{10}I_4$$

Both D_{21} and D_{22} are 4×4 matries; hence Eq. (76) consists of four ordinary equations.

Equations (38) and (39) linearized and transformed to matrices become

$$\begin{aligned}&[G_2(X_0^*) + G_1(X_0)J]v_{\delta X} = 0\\ &[G_2(B_0^*) + G_1(B_0)J]v_{\delta B} = 0\end{aligned} \tag{77}$$

From the properties of G_1 and G_2, Eqs. (77) can be simplified. Moreover, these matrix equations are equivalent to

$$v_{X0}^T v_{\delta X} = 0, \qquad v_{B0}^T v_{\delta B} = 0 \tag{78}$$

respectively. Equations (78) corroborate once more the well-known result that correction has to be orthogonal to the initial approximation.

Combining Eqs. (74), (76), and (78) we obtain a set of nine linear algebraic equations, written in block form as

$$\begin{bmatrix} D_{11} & D_{12} & 0 & v_{B0} \\ D_{21} & D_{22} & v_{X0} & 0 \\ 0 & v_{X0}^T & 0 & 0 \\ v_{B0}^T & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_{\delta B} \\ v_{\delta X} \\ \delta\gamma_1 \\ \delta\gamma_2 \end{bmatrix} = \begin{bmatrix} -v_{P10} \\ -v_{P20} \\ 0 \\ 0 \end{bmatrix} \tag{79}$$

The obtained set of ordinary linear algebraic equations is very suitable for computations due to the simple structure of matrices G_1 and G_2 and consequently matrices D. The structure of these matrices also gives convenient conditions for checking the calculations. The desired corrections are easily found by solving Eq. (79).

IV. Conclusions

We have shown that the application of quaternions provides a brief and convenient way of solving the spacecraft orientation problem. The formulation of the initial problem in terms of the conditional quaternion extremum problem makes it possible to obtain a solution in a simple manner with no numerous trigonometric calculations, even when nonzero daily parallax objects are observed. The quaternion method is very suitable both for theoretical analysis and computations.

We gave only a theoretical ground for the quaternion algorithm and confirmed it by means of comparison with the well-known theoretical results. Very important problems of numerical realization of this algorithm (how quickly the algorithm will converge, how near to "the truth" the initial estimate must be for convergence to be achievable, whether the algorithm may have singularity problems for small parallax effects, etc.) are of great interest and require additional investigation for different practical cases.

The considered quaternion algorithm seems to be also useful for solving many different problems in space photogrammetry, astronomy, and control of a spinning space bodies, particularly if a strapdown navigation system is used.

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References

[1]Wahba, G., "A Least Squares Estimate of Satellite Attitude, Problem 65.1," *SIAM Review*, Vol. 7, No. 3, 1965, p. 409.

[2]Lerner, G. M., "Three-Axis Attitude Determination," *Spacecraft Attitude Determination and Control*, edited by J. R. Wertz, Reidel, Dordrecht, The Netherlands, 1978, pp. 420–428.

[3]Shuster, M. D., "A Survey of Attitude Representations," *Journal of the Astronautical Sciences*, Vol. 41, No. 4, 1993, pp. 439–517.

[4]Shuster, M. D., and Oh, S. D., "Three-Axis Attitude Determination from Vector Observations," *Journal of Guidance and Control*, Vol. 4, No. 1, 1981, pp. 70–77.

[5]Shuster, M. D., "Maximum Likelihood Estimation of Spacecraft Attitude by Vector Observations," *Journal of the Astronautical Sciences*, Vol. 37, No. 1, 1989, pp. 79–88.

[6]Shuster, M. D., "A Simple Kalman Filter and Smoother for Spacecraft Attitude," *Journal of the Astronautical Sciences*, Vol. 37, No. 1, 1989, pp. 89–106.

[7]Markley, F. L., "Attitude Determination Using Vector Observations and the Singular Value Decomposition," *Journal of the Astronautical Sciences*, Vol. 36, No. 3, 1988, pp. 245–258.

[8]Markley, F. L., "Parameterization of the Attitude," *Spacecraft Attitude Determination and Control*, edited by J. R. Wertz, Reidel, Dordrecht, The Netherlands, 1978, pp. 410–420.

[9]Golub, G. H., and von Matt, U., "Quadratically Constrained Least Squares and Quadratic Problems," *Numerische Mathematik*, Vol. 59, 1991, pp. 561–580.

[10]Gienger, G., "Single Axis Attitude Determination by the Linear Least Squares Method with Norm Constrained," *Spaceflight Dynamics 1993*, edited by J. Teles and M. V. Samii, Vol. 84, Advances in the Astronautical Sciences, American Astronomical Society, San Diego, CA, 1993, pp. 729–743 (AAS 93-300).

[11]Bar-Itzhack, I. Y., and Oshman, J., "Attitude Determination from Vector Observations: Quaternion Estimation," Department of Aeronautical Engineering Technical Report, Technion Israel Institute of Technology, Jan. 1982.

[12]Sanso, F. "An Exact Solution of the Roto-Translation Problem," *Photogrammetria*, Vol. 29, 1973, pp. 203–216.

[13]Urmaev, M. S., *Space Photogrammetry*, Nedra, Moscow, 1989, pp. 42–67 (in Russian).

[14]Branez, V. N., and Shmiglevsky, I. P., *Quaternions in Rigid Body Orientation Problems*, Nauka, Moscow, 1973 (in Russian).

[15]Grafarend, E. W., and Schaffrin, B., "Vectors, Quaternions and Spinors: A Discussion of Algebras Underlying 3-Dimensional Geodesy," *Feestbundel ter Gelegnhid van de 65ste Verjaardag van Professor Baarda, Deel 1*, Geodetic Computer Centre (LGR), Delft, 1983, pp. 111–134.

[16]Battin, R. H., "New Properties of the Rotation Matrix," International Astronautical Federation, New York, Paper IAF-87-344.

[17]Aramanovitch, L. I., "Solution of the Problem of Photo Orientation in Aerial Surveying by Means of Quaternion Algebra," *Manuscripta Geodaetica*, Vol. 17, 1992, pp. 334–341.

[18]Aramanovitch, L. I., "Application of Quaternions to Photo Relative Orientation Problems," *Izvestia Vuzov, Geodesy and Aerophotography*, Vol. 4, 1990, pp. 99–110 (in Russian).

[19]Aramanovitch, L. I., "On the Application of Quaternions for Determination of Spacecraft Initial Attitude and Velocity in Strap-Down Navigation," *Manuscripta Geodaetica*, Vol. 17, 1992, pp. 342–350.

[20]Aramanovitch, L. I., "Quaternion Non-Linear Filter for Estimation of Rotating Body Attitude," *Mathematical Methods in the Applied Sciences*, Vol. 18, 1995 (to be published).

[21]Liusternik, L. A., and Sobolev, V. I., *Elements of Functional Analysis*, Nauka, Moscow, 1965.